A Series of Cubes

Submission deadline: August 31^{st} 2018

Evaluate the infinite series

$$1 - \frac{2^3}{1!} + \frac{3^3}{2!} - \frac{4^3}{3!} + \frac{5^3}{4!} - \cdots$$

The problem was solved by

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Discussion:

Since

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

we have that

$$xe^{-x} = x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!} - \cdots$$

By differentiating above we get

$$e^{-x} - xe^{-x} = 1 - 2x + \frac{3x^2}{2!} - \frac{4x^3}{3!} + \frac{5x^4}{4!} - \cdots$$

Multiply the above by x to get

$$xe^{-x} - x^2e^{-x} = x - 2x^2 + \frac{3x^3}{2!} - \frac{4x^4}{3!} + \frac{5x^5}{4!} - \cdots$$

By differentiating the above we get

$$e^{-x}(1 - 3x + x^2) = 1 - 2^2x + \frac{3^2x^2}{2!} - \frac{4^2x^3}{3!} + \frac{5^2x^4}{4!} - \cdots$$

Multiply above by x to get

$$e^{-x}(x - 3x^2 + x^3) = x - 2^2x^2 + \frac{3^2x^3}{2!} - \frac{4^2x^4}{3!} + \frac{5^2x^5}{4!} - \cdots$$

Differentiate the equation above and we get

$$e^{-x}(1 - 6x + 3x^2) - e^{-x}(x - 3x^2 + x^3) = 1 - 2^3x + \frac{3^3x^2}{2!} - \frac{4^3x^3}{3!} + \frac{5^3x^4}{4!} - \cdots$$

Let $x = 1$ in the equation above to get

$$-e^{-1} = 1 - 2^3 + \frac{3^3}{2!} - \frac{4^3}{3!} + \frac{5^3}{4!} - \cdots$$