Derivatives.

Submission deadline: November 28^{th} 2020

Let $p(x)$ be a monic polynomial of degree $m \geq 1$. Let D^n denote $\frac{d^n}{dx^n}$. Show that

$$e^{p(x)} D^n \big(e^{-p(x)}\big)$$

is a polynomial and find its degree. Find the ratio of the leading coefficient to the constant term.

The problem was solved by

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Discussion

Below, we give some ideas and the interested reader can construct a rigorous proof using mathematical induction.

Let

$$f_k(x) = e^{-p(x)} q_k(x)$$

where k is an integer and $q_k(x)$ is a polynomial. Then, by differentiating once

$$f_k'(x) = e^{-p(x)}(q_k'(x) - q_k(x) \cdot p'(x))$$

Now let

$$q_{k+1}(x) = q_k'(x) - q_k(x) \cdot p'(x). \tag{1}$$

Then, it is easy to see that

$$deg(q_{k+1}) = deg(q_k) + (m-1) \tag{2}$$

We may take $e^{-p(x)} = e^{-p(x)} q_0(x)$, where $q_0(x) = 1$, hence $deg(q_0) = 0$. Now it is clear that $D^n(e^{-p(x)} q_0(x)) = e^{-p(x)} q_n(x)$, where $q_n(x)$ is a polynomial, and $e^{p(x)} D^n(e^{-p(x)}) = q_n(x)$. From (2) it is clear that $deg(q_n(x)) = n \cdot (m-1)$. Thus, degree of $e^{p(x)} D^n(e^{-p(x)})$ is

$$n \cdot (m-1)$$

If l_k denotes the leading coefficient of q_k, then from (1) it is clear that

$$l_{k+1} = (-m) l_k$$

and since $l_0 = 1$, it easily follows that l_n is $(-m)^n$.

For ceratin $p(x)$, the constant term could be 0, thus it appears to be that the problem was too ambitious in trying to find the ratio.