Onto the Natural Numbers

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Find a polynomial $p(x, y)$ such that as m and n go through all positive integer values, $p(m, n)$ takes on all positive integer values once and only once.

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Discussion

Consider the lattice points (m, n) in the XY plane with positive integer coordinates. Assign the value 1 to the point $(1, 1)$. Next consider the two closest points to (1,1), which are (2,1) and (1,2). Join them by a straight line. Starting from the bottom of the line assign successive positive integer values to lattice points on the line. Thus, (2,1) is assigned 2 and (1,2) is assigned 3. Continuing in a similar fashion, next 3 points (3,1), (2,2),(1,3) get assigned to numbers 4, 5, 6. It is clear that each positive integer is assigned to a lattice point only once if we continue a similar assignment.

Now, count the number of lattice points along straight lines from bottom to top until the point (m, n) is reached. Then it can be seen that there are

$$\frac{(m+n-2)(m+n-1)}{2} + n$$

points. Now we can guess the polynomial. Let

$$p(x, y) = \frac{(x+y-2)(x+y-1)}{2} + y$$

Now we need to prove that $p(x, y)$ satisfies the given conditions.

If m, n are positive integers, then $(m+n-2)(m+n-1)$ is a non-negative even integer, hence $p(m, n)$ is a positive integer.

Also notice that if $m + n \geq 2$, then $p(m, n)$ increases whenever m or n increases. Thus, $p(m, n)$ cannot repeat values as m and n go through positive integers.

Given a positive integer $k > 1$, first find the largest integer r so that

$$r(r+1)/2 < k.$$

Let $n = k - r(r+1)/2$ and $m = r + 2 - n$. Then $p(m, n) = k$.