Multiples of 5

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Describe all positive integers n such that $n^5 - n$ is divisible by 5.

The problem was solved by

- Rohan Mitra, *American University of Sharjah, UAE.*

- Mümtaz Ulaş KESKİN, *Erciyes University.*

- Parv Bhadra, *Grade 11, GEMS Modern Academy, Dubai, UAE.*

- Ahmet Yüksel, *Aydın,Turkey.*

- Sandupama Gunawardana, *Hillwood College, Kandy, Sri Lanka.*

- Hari Kishan, *(Ex.) Department of Mathematics, D.N. College, Meerut, India.*

- Atakan Erdem, *Middle East Technical University (METU), Ankara, Turkey.*

- Muhammed YÜKSEL, *Ankara, Turkey.*

- Merdangeldi Bayramov, *Ashgabat, Turkmenistan.*

Discussion;

First assume that $n = 2k$ for some positive integer k.

It is easy to see that $n^5 - n = 2k \cdot (2k-1) \cdot (2k+1) \cdot (4k^2+1)$. Thus, $n^5 - n = 2k \cdot (2k-1) \cdot (2k+1) \cdot (4k^2 - 4 + 5)$. Therefore,$n^5 - n$ is equal to,

$$2k \cdot (2k-1) \cdot (2k+1) \cdot (2k+2) \cdot (2k-2) + 5 \cdot 2k \cdot (2k-1) \cdot (2k+1)$$

In the summation above, the first term is a product of five consecutive integers, hence is divisible by 5. The second term is clearly divisible by 5. Thus, when n is even, $n^5 - n$ is divisible by 5.

Next assume that $n = 2k+1$ for some positive integer k. Clearly $n^5 - n = (2k+1) \cdot (2k) \cdot (2k+2) \cdot (4k^2 + 4k + 2)$. Thus

$$n^5 - n = (2k+1) \cdot (2k) \cdot (2k+2) \cdot ((2k+3)(2k-1) + 5)$$

Therefore,

$$n^5 - n = (2k+1) \cdot (2k) \cdot (2k+2) \cdot (2k+3) \cdot (2k-1) + 5(2k+1) \cdot (2k) \cdot (2k+2)$$

The first term on the right hand side is a product of 5 consecutive integers, therefore is divisible by 5 and second term is clearly divisible by 5. Hence when n is odd, $n^5 - n$ is divisible by 5.

Therefore $n^5 - n$ is divisible by 5 for all integers n.