Less then Average

Submission deadline: July 28^{th} 2024

Let T be a triangle with sides a, b and c. Assume that c is less than the average of the other two sides. Prove that the angle opposite to the side c is less than the average of the other two angles.

The problem was solved by

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Discussion:

Let γ denote the angle opposite to the side c and denote the other two angles by α and β. Then, we have that

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

Since $(a+b)/2 \geq c$, it follows that

$$\frac{(a+b)^2}{4} \geq a^2 + b^2 - 2ab\cos(\gamma)$$

Further simplification of the inequality above yields,

$$2ab(1 + 4\cos(\gamma)) \geq 3(a^2 + b^2)$$

But $a^2 + b^2 \geq 2ab$, therefore,

$$2ab(1 + 4\cos(\gamma)) \geq 6ab.$$

Thus,

$$\cos(\gamma) \geq \frac{1}{2}$$

Note that $\cos(x)$ decreases when $0^\circ < x < 180^\circ$, and $\cos(60^\circ) = 1/2$, hence

$$\gamma \leq 60^\circ$$

Thus, $\alpha + \beta \geq 120^\circ$ and we obtain the desired result.