Ascending and Descending

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Let n be a three digit natural number with at least 2 distinct digits (leading zeros are allowed). Consider the following iterative process;

1. Arrange the digits of n in descending and then in ascending order to get two natural numbers, adding leading zeros if necessary.
2. Subtract the smaller number from the bigger number.
3. Go back to step 1 and repeat the process with the natural number obtained in step 2.

Prove that for any n as described above, the process above yields the same number after finitely many iterations.

The problem was solved by

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Discussion:

Let $A(n)$ be the number obtained by arranging digits of n is ascending order and $D(n)$ be the number obtained by arranging digits of n in descending order. If $A(n) = cba$, then $D(n) = abc$. Thus,

$$D(n) - A(n) = 100a + 10b + c - (100c + 10b + a)$$

Hence, $D(n) - A(n) = (a - c) \cdot 99$. Clearly, $a - c$ is a natural number between 1 and 9. Therefore, $D(n) - A(n)$ can only take the following 9 values; 099, 198, 297, 396, 495, 594, 693, 792 and 891. It is easy to see that

$$D(495) - A(495) = 495.$$

Thus, if the process for n at some point yields 495, then it will continue to yield 495 for all subsequent iterations.

Next we apply the process to the remaining 8 numbers listed above. However, if m and n consist of same digits but in different order, it is easy to see that the process yields identical results. Thus, it suffices to apply the process only to 099, 198, 297, 396 out of the 8 numbers above.

$n = 396$; since $D(396) - A(396) = 594$, it is clear that in two iterations 495 will be reached.

$n = 297$; since $D(297) - A(297) = 693$, it is clear that in three iterations 495 will be reached.

$n = 198$; since $D(198) - A(198) = 792$, it is clear that in four iterations 495 will be reached.

$n = 099$; since $D(099) - A(099) = 891$, it is clear that in five iterations 495 will be reached.

Thus, for any three digit number with at least 2 distinct digits, the process will yield 495 in at most six iterations.

We encourage the interested readers to look up the Kaprekar's routine.